

# The pressure of a weakly magnetized deconfined QCD matter within one-loop Hard-Thermal-Loop perturbation theory

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Aritra Bandyopadhyay<sup>a</sup>, Najmul Haque<sup>b</sup> and Munshi G. Mustafa<sup>a</sup>

<sup>a</sup>*Theory Division, Saha Institute of Nuclear Physics, HBNI, 1/AF Bidhan Nagar, Kolkata 700064, India.*

<sup>b</sup>*Institut für Theoretische Physik, Justus-Liebig-Universität Giessen, 35392 Giessen, Germany*

*E-mail:* [aritra.bandyopadhyay@saha.ac.in](mailto:aritra.bandyopadhyay@saha.ac.in),  
[Najmul.Haque@theo.physik.uni-giessen.de](mailto:Najmul.Haque@theo.physik.uni-giessen.de),  
[munshigolam.mustafa@saha.ac.in](mailto:munshigolam.mustafa@saha.ac.in)

**ABSTRACT:** We compute an analytic expression for the pressure of a weakly magnetized deconfined QCD matter within one-loop Hard-Thermal-Loop perturbation theory (HTLpt) at finite temperature and chemical potential. We also discuss the modification of QCD Debye mass of such matter for an arbitrary magnetic field. It is found to exhibit some interesting features depending upon the three different scales, *i.e.*, the quark mass, temperature and the strength of the magnetic field. We also obtain an analytic expression for Debye mass in a weak field approximation. The various divergences appearing in the quark and gluon free energies are regulated through appropriate counter terms. The low temperature behaviour of the pressure is found to strongly depend on the magnetic field than that at high temperature. We also discuss the specific problem with one-loop HTLpt associated with the over-counting of certain orders in coupling.

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## 1 Introduction

Quark Gluon Plasma (QGP) is a thermalized color deconfined state of nuclear matter in the regime of Quantum Chromo Dynamics (QCD) under extreme conditions such as very high temperature and/or density. For the past couple of decades, different high energy Heavy-Ion-Collisions (HIC) experiments are under way, *e.g.* RHIC @ BNL, LHC @ CERN and upcoming FAIR @ GSI, to study this novel state of QCD matter within the largely unknown QCD phase diagram. In recent years the focus has also shifted towards the noncentral HIC, where a very strong magnetic field is created in the direction perpendicular to the reaction plane due to the spectator particles that are not participating in the collisions [1–5]. Recent

experimental evidences of photon anisotropy provided by the PHENIX Collaboration [6] has also challenged the present theoretical tools. By assuming a presence of large anisotropic magnetic field generated in HIC, eventually some explanations were made [7] in support of those experimental findings. In fact this has prompted that a theoretical study is much needed by considering the effects of intense background magnetic field on various aspects and observables of noncentral HIC. Also some of these studies have subsequently revealed that the strong magnetic field generated during the noncentral HIC is also time dependent. More specifically, it rapidly decreases with time [8, 9]. Nevertheless, the inclusion of an external magnetic field in QGP introduces also an extra energy scale in the system. At the time of the noncentral HIC, the value of the created magnetic field  $B$  is very high compare to the temperature  $T$  ( $T^2 < q_f B$  where  $q_f$  is the absolute charge of the quark with flavor  $f$ ) associated with the system. It is estimated upto the order of  $q_f B \sim 15m_\pi^2$  in the LHC at CERN [10]. On the other hand, neutron stars (NS), or more specifically magnetars are also known to possess strong enough magnetic field [11–13]. In this regime of study one usually works in the strong magnetic field approximation [14].

Now, the presence of an external anisotropic field in the medium calls for the appropriate modification of the present theoretical tools to investigate various properties of QGP and a numerous activity is in progress. Over the last few years, several novel phenomena came into light, *e.g.*, chiral magnetic effect [15–17], magnetic catalysis [18–20] and inverse magnetic catalysis [21–27] at finite temperature; chiral- and color-symmetry broken/restoration phase [28–31]; thermodynamic properties [30–32]; refractive indices and decay constant [33, 34] of mesons in hot magnetised medium; soft photon production from conformal anomaly [7, 35] in HIC; modification of dispersion properties in a magnetised hot QED medium [36]; synchrotron radiation [37], dilepton production from a hot magnetized QCD plasma [14, 37–40] and in strongly coupled plasma in a strong magnetic field [41].

Now, the QCD equation of state (EoS) is a generic quantity and of phenomenological importance for studying the hot and dense matter, QGP, created in HIC. At zero chemical potential and finite temperature Lattice QCD (LQCD) established itself as the most reliable method to calculate thermodynamic functions. Unfortunately at finite chemical potential LQCD faces the infamous sign problem. Information about the thermodynamic functions in LQCD can still be extracted by making a Taylor expansion of the partition function around zero baryonic chemical potential and extrapolating the result [42]. But due to the finite number of Taylor coefficients such extrapolation has it's own limitations. On the other hand, naively, the asymptotic freedom of QCD leads us to expect that bare perturbation theory should be a reliable guide to calculate these properties of matter at high temperature and/or high density [43–49]. Although, it has been recognized early on that this is not so. Technically, infrared divergences plague the calculation of observables at finite temperature, preventing the determination of high order corrections. In order to cope with this difficulty, whose origin is the presence of massless particles, it has been suggested to reorganize perturbation theory, by performing the expansion around of a system of massive quasiparticles. The motivation for doing so is that thermal fluctuations can generate a mass. It amounts to a resummation of a class of loop diagrams, where the loop momenta are of the order of the temperature. Such diagrams are those which

contribute to give the excitations a thermal mass. The Hard Thermal Loop perturbation theory (HTLpt) is one such state-of-the-art resummed perturbation theory [50]. In HTLpt the EOS of QCD *in absence of magnetic field* has systematically been computed within one-loop(Leading order (LO)) [50–57], two loop (next-LO (NLO)) [58–61] and three loop (next-to-NLO (NNLO)) [62–68] at finite temperature and chemical potential. Though the all loop order calculations are gauge invariant, the three loop results are complete in  $g^5$ , fully analytic that does not require any free fit parameter beside renormalization scale. However, no work has been done to study the QCD EoS within HTLpt in presence of an external magnetic field, though some thermodynamic properties of low lying hadrons within chiral perturbation theory in presence of magnetic field are studied in recent years [30–32]. Also the thermomagnetic correction to the quark-gluon vertex in the presence of a weak magnetic field within the HTL approximation is computed recently [69].

In view of this, presently, a systematic determination of EOS for magnetized hot QCD medium is of great importance. In this article we make an effort to derive the pressure of a magnetized hot and dense deconfined QCD medium. Usually two kinds of approaches are taken in all the previous studies of EoS in presence of magnetic field. In first kind the pressure remains isotropic and the system can be easily described in terms of standard thermodynamic relations [12, 13]. In other one, the breaking of the spherical symmetry due to the anisotropic background magnetic field in a preferred direction [70–74] is taken into consideration. Subsequently, this results in an anisotropic pressure arising from the difference between pressure components that are transverse to and longitudinal to the background magnetic field direction. Eventually, the difference in stress causes the deformation of the fireball produced in heavy ion collisions or the NS. There are also some recent LQCD calculations, which incorporate both of these schemes [75]. However, it is also shown in [32], that pressure anisotropy decreases with the increase in temperature. Moreover, the magnetic field created in noncentral HIC is a fast decreasing function of time [8, 9], it is expected that by the time the quarks and gluons thermalize in a QGP medium, the magnetic field strength becomes sufficiently weak. By virtue of which temperature at that time becomes the largest energy scale of the system. In this regime, in principle, one can work within the weak magnetic field approximation which, in addition, leads to analytical simplicity. Because of this, throughout our calculation, we consider the external background magnetic field to be a weaker than the associated temperature of the system. We here consider the first approach, i.e., to assume that the magnetic pressure adds isotropically to the total pressure. In this article, as a first effort, we compute the pressure of a hot and dense deconfined QCD medium in presence of a weak magnetic field using one-loop HTLpt

The paper has been organized as follows: in section 2 the HTLpt has briefly been outlined. In sections 3 and 4, respectively, the quark propagator and the HTL self-energy have been discussed in presence of weak magnetic field. The QCD Debye mass has been obtained in section 5 for an arbitrary magnetic field and also in weak field approximation. The total free energy and the pressure of a hot and dense but weakly magnetized deconfined QCD matter are obtained in section 6. Finally, we conclude in section 7.

## 2 Hard Thermal Loop perturbation theory

The HTLpt Lagrangian density can be written as a rearrangement of the in-medium perturbation theory for QCD. It reads as

$$\mathcal{L} = (\mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{HTL}})|_{g \rightarrow \sqrt{\delta}g} + \Delta\mathcal{L}_{\text{HTL}}, \quad (2.1)$$

where the added HTL term is [59]

$$\mathcal{L}_{\text{HTL}} = (1 - \delta)im_q^2 \bar{\psi} \gamma^\mu \left\langle \frac{x_\mu}{x \cdot D} \right\rangle_x \psi - \frac{1}{2}(1 - \delta)m_D^2 \text{Tr} \left( G_{\mu\alpha} \left\langle \frac{x^\alpha x_\beta}{(x \cdot D)^2} \right\rangle_x G^{\mu\beta} \right), \quad (2.2)$$

where  $x^\mu = (1, \hat{\mathbf{x}})$  is a light-like four-vector. The angular bracket indicates an average over the direction of the three dimensional unit vector  $\hat{\mathbf{x}}$ .  $m_D$  and  $m_q$  can be recognized as the Debye screening mass and the thermal quark mass, respectively. They account for the screening effects, which we shall also discuss in section 5.

A HTLpt is formulated by treating  $\delta$  as a formal expansion parameter. As we can see from Eq. (2.2) and Eq. (2.1), the HTLpt Lagrangian reduces to the QCD Lagrangian if we set  $\delta = 1$ . In HTLpt physical observables are calculated first by expanding in powers of  $\delta$ , then truncating at some specified order, and finally setting  $\delta = 1$ . For example, to obtain the one-loop or the leading order (LO) results, one has to expand the corresponding observable up to order  $\delta^0$ . Here we also note that HTLpt is gauge invariant order-by-order in the  $\delta$  expansion. So, eventually, the results obtained are independent of the gauge-fixing parameter.

The total thermodynamic free energy up to one-loop order in presence of a constant weak magnetic field,  $B$ , can be obtained by adding the gluonic and the fermionic contribution to the tree level contribution due to the constant magnetic field  $F_0$  as

$$F^{\text{HTL}} = F_q^{\text{HTL}} + F_g^{\text{HTL}} + F_0 + \Delta\mathcal{E}_0, \quad (2.3)$$

where  $F_0 = \frac{1}{2}B^2$  and the HTL counter term is given [60] as

$$\Delta\mathcal{E}_0 = \frac{d_A}{128\pi^2\epsilon} m_D^4, \quad (2.4)$$

with  $d_A = N_c^2 - 1$ ,  $N_c$  is the number of color in fundamental representation.

In statistical field theory the partition function  $Z$  can be represented as a functional determinant and by which the fermionic (quark) part of the QCD free energy in one-loop order can be written as

$$F_q^{\text{HTL}} = -N_c N_f \int \frac{d^4 P}{(2\pi)^4} \ln (\det [S_{\text{eff}}^{-1}(P)]), \quad (2.5)$$

where  $P \equiv (p_0, \mathbf{p})$  is the four momentum of the external fermion. The effective or the dressed fermion propagator  $S_{\text{eff}}(P)$  can be written as

$$S_{\text{eff}}(P) = \frac{1}{\not{P} - m_f - \Sigma(P)}, \quad (2.6)$$

where  $m_f$  is the mass of the fermion of flavor  $f$  and  $\Sigma(P)$  is the fermion self-energy in presence of a constant weak magnetic field within HTL approximation.

On the other hand, the gluonic contribution for the one-loop HTL free energy reads as

$$F_g^{HTL} = (N_c^2 - 1) [(d - 1)F_g^T + F_g^L], \quad (2.7)$$

where  $d$  is the spatial dimension of the theory and we use  $d = 3 - 2\epsilon$  for dimensional regularization. Also,  $F_g^T$  and  $F_g^L$  are, respectively, the transverse and longitudinal part of the gluon free energy given by

$$F_g^T = \frac{1}{2} \int \frac{d^4 Q}{(2\pi)^4} \ln(Q^2 + \Pi_T), \quad (2.8)$$

$$F_g^L = \frac{1}{2} \int \frac{d^4 Q}{(2\pi)^4} \ln\left(q^2 + \frac{q^2}{Q^2} \Pi_L\right). \quad (2.9)$$

Here  $Q \equiv (q_0, q)$  is the external gluon four momentum and  $\Pi_T$  and  $\Pi_L$  are, respectively, the transverse and the longitudinal part of the gluon self-energy, both of which can be expressed in terms of the Debye screening mass  $m_D$  (e.g., see Appendix A).

### 3 Fermion propagator in presence of a weak magnetic field

For obtaining the fermion self-energy in presence of a weak magnetic field one needs to modify the fermion propagator, which is briefly outline here. Assuming a constant magnetic field pointing towards the  $z$  direction ( $\vec{B} = B\hat{z}$ ), the charged fermion propagator can be described in the symmetric gauge via the Schwinger proper time formalism [76]. In coordinate space it is represented as

$$S(X, X') = e^{\Phi(X, X')} \int \frac{d^4 K}{(2\pi)^4} e^{-iK(X-X')} S(K), \quad (3.1)$$

where  $S(K)$  is the translation and gauge invariant part of the fermion propagator in a background gauge potential. The phase factor  $\Phi$  is responsible for breaking of gauge and translation invariance, and explicit form of  $\Phi$  is irrelevant as it drops out in a gauge invariant calculation. The Schwinger propagator  $S(K)$  in momentum space can be written [76] as an integral over proper time  $s$  by

$$\begin{aligned} iS(K) &= \int_0^\infty ds \exp \left[ is \left( K_\parallel^2 - m_f^2 - \frac{K_\perp^2}{q_f B s} \tan(q_f B s) \right) \right] \\ &\times [(\not{K}_\parallel + m_f) (1 + \gamma_1 \gamma_2 \tan(q_f B s)) - \not{K}_\perp (1 + \tan^2(q_f B s))]. \end{aligned} \quad (3.2)$$

Here,  $q_f$  is the absolute charge of the fermion of flavor  $f$ . We consider two light flavours ( $N_f = 2$ ) consisting of  $u$  and  $d$  quarks of equal current quark mass ( $m_f = m_u = m_d = 5$  MeV if not said otherwise). We outline the notation used in (3.2) and are followed throughout below as

$$a^\mu = a_\parallel^\mu + a_\perp^\mu; \quad a_\parallel^\mu = (a^0, 0, 0, a^3); \quad a_\perp^\mu = (0, a^1, a^2, 0),$$

$$g^{\mu\nu} = g_{\parallel}^{\mu\nu} + g_{\perp}^{\mu\nu}; \quad g_{\parallel}^{\mu\nu} = \text{diag}(1, 0, 0, -1); \quad g_{\perp}^{\mu\nu} = \text{diag}(0, -1, -1, 0),$$

$$(a \cdot b) = (a \cdot b)_{\parallel} - (a \cdot b)_{\perp}; \quad (a \cdot b)_{\parallel} = a^0 b^0 - a^3 b^3; \quad (a \cdot b)_{\perp} = (a^1 b^1 + a^2 b^2),$$

where  $\parallel$  and  $\perp$  are, respectively, the parallel and perpendicular components, now separated out due to the anisotropy of the external magnetic field.

In the rest part of the paper we will consider the external magnetic field to be a weak one. In this weak field approximation ( $T^2 > q_f B$ ) the fermion propagator in the momentum space takes the simplified form [69] following the Taylor expansion of Eq. (3.2) and keeping terms up to  $\mathcal{O}(q_f B)$  as

$$S(K) = \frac{\not{K} + m_f}{K^2 - m_f^2} + i\gamma_1 \gamma_2 \frac{\not{K}_{\parallel} + m_f}{(K^2 - m_f^2)^2} (qB) + \mathcal{O}[(q_f B)^2] \quad (3.3)$$

$$= S_1(K) + S_2(K) + \mathcal{O}[(q_f B)^2]. \quad (3.4)$$

As seen the fermionic propagator of  $\mathcal{O}[q_f B]$  in the weak field approximation gets decomposed into a tree level one (i.e.,  $S_1(K)$ ) without involving magnetic field and a one (i.e.,  $S_2(K)$ ) that involves correction due to the presence of magnetic field.

#### 4 HTL Quark self-energy in presence of a weak magnetic field

Using the modified fermion propagator in weak field approximation in Eq. (3.4) one can straightforwardly write down the quark self-energy in Feynman gauge from Fig. 1 as

$$\begin{aligned} \Sigma(P) &= -ig^2 C_F \int \frac{d^4 K}{(2\pi)^4} \gamma_{\mu} S(K) \gamma^{\mu} \Delta(P - K), \\ &= -ig^2 C_F \int \frac{d^4 K}{(2\pi)^4} \gamma_{\mu} (S_1(K) + S_2(K)) \gamma^{\mu} \Delta(P - K), \\ &= \Sigma_1(P) + \Sigma_2(P), \end{aligned} \quad (4.1)$$

where  $C_F = (N_c^2 - 1)/(2N_c)$  and the gluonic propagator is given as

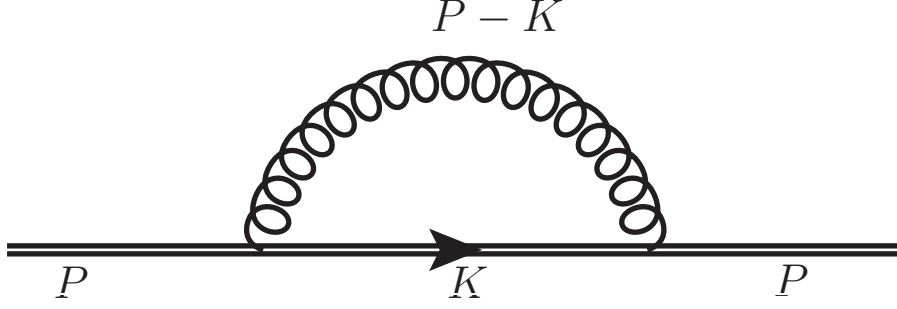
$$\Delta(P - K) = \frac{1}{(p_0 - k_0)^2 - (p - k)^2} = \frac{1}{(p_0 - k_0)^2 - E_{pk}^2}. \quad (4.2)$$

We note that  $\Sigma_1(P)$  is the quark self-energy that does not pick up any magnetic field contribution whereas  $\Sigma_2(P)$  is the one that involves the correction in presence of magnetic field.

##### 4.1 Evaluation of $\Sigma_1(P)$

Now, the in-medium tree level self-energy  $\Sigma_1(P)$  can be written from Eq. (4.1) as

$$\begin{aligned} \Sigma_1(P) &= -ig^2 C_F \int \frac{d^4 K}{(2\pi)^4} \gamma_{\mu} \frac{(\not{K} + m_f)}{(K^2 - m_f^2)} \gamma^{\mu} \Delta(P - K) \\ &= -ig^2 C_F \int \frac{d^4 K}{(2\pi)^4} \gamma_{\mu} \not{K} \gamma^{\mu} \tilde{\Delta}(K) \Delta(P - K) \end{aligned}$$



**Figure 1.** Self energy diagram for a quark in weak magnetic field approximation. The double line indicates the modified quark propagator in presence of magnetic field.

$$= -2g^2 C_F \sum_{\{K\}} \not{K} \tilde{\Delta}(K) \Delta(P - K), \quad (4.3)$$

where the fermionic propagator is written as

$$\tilde{\Delta}(K) = \frac{1}{k_0^2 - (k^2 + m_f^2)} = \frac{1}{k_0^2 - E_k^2}, \quad (4.4)$$

and we have neglected the  $m_f$  in the numerator by the virtue of HTL approximation ( $T \gg m_f$ ). Also at finite temperature, the loop integration measure is replaced by

$$\int \frac{d^4 K}{(2\pi)^4} \longrightarrow i \sum_{\{K\}} = iT \sum_{k_0} \int \frac{d^3 k}{(2\pi)^3}. \quad (4.5)$$

Now in Eq. (4.3) there are two frequency sums:  $T \sum_{k_0} \tilde{\Delta}(K) \Delta(P - K)$  and  $T \sum_{k_0} k_0 \tilde{\Delta}(K) \Delta(P - K)$ . We evaluate them by using the mixed representation prescription [77]. In the mixed representation the fermionic and bosonic propagators are, respectively, written as

$$\tilde{\Delta}(K) = \frac{1}{k_0^2 - E_k^2} = \int_0^\beta d\tau_1 e^{k_0 \tau_1} \Delta_F(\tau_1, E_k), \quad (4.6)$$

$$\Delta(P - K) = \frac{1}{(p_0 - k_0)^2 - E_{pk}^2} = \int_0^\beta d\tau_2 e^{(p_0 - k_0) \tau_2} \Delta_B(\tau_2, E_{pk}), \quad (4.7)$$

with

$$\Delta_F(\tau_1, E_k) = \frac{1}{2E_k} \left[ (1 - n_F(E_k)) e^{-E_k \tau_1} - n_F(E_k) e^{E_k \tau_1} \right], \quad (4.8)$$

$$\Delta_B(\tau_2, E_{pk}) = \frac{1}{2E_{pk}} \left[ (1 + n_B(E_{pk})) e^{-E_{pk} \tau_2} + n_B(E_{pk}) e^{E_{pk} \tau_2} \right]. \quad (4.9)$$

Using these now the sum can be performed as

$$T \sum_{k_0} \tilde{\Delta}(K) \Delta(P - K) = \int_0^\beta d\tau e^{p_0 \tau} \Delta_F(\tau, E_k) \Delta_B(\tau, E_{pk}), \quad (4.10)$$



$$= - \sum_{l,r=\pm 1} \frac{r [1 - n_F(rE_k)] [1 + n_B(lE_{pk})]}{4E_k E_{pk} (p_0 - rE_k - lE_{pk})} \left[ e^{-\beta(rE_k + lE_{pk})} + 1 \right]. \quad (4.11)$$

We note that only two terms out of the available four are leading order in  $T$  for  $r = -l$ . Combining those two terms one gets

$$T \sum_{k_0} \tilde{\Delta}(K) \Delta(P - K) = \frac{n_F(E_k) + n_B(E_{pk})}{4E_k E_{pk}} \left( \frac{1}{p_0 + E_k - E_{pk}} - \frac{1}{p_0 - E_k + E_{pk}} \right). \quad (4.12)$$

Following the similar procedure for the second sum we get

$$T \sum_{k_0} k_0 \tilde{\Delta}(K) \Delta(P - K) = - \frac{n_F(E_k) + n_B(E_{pk})}{4E_{pk}} \left( \frac{1}{p_0 + E_k - E_{pk}} + \frac{1}{p_0 - E_k + E_{pk}} \right). \quad (4.13)$$

Now, we use the following relations within Hard Thermal Loop approximation:

$$n_B(E_{pk}) \simeq n_B(E_k), \quad (4.14)$$

$$E_{pk} \simeq k - \vec{p} \cdot \hat{k}, \quad (4.15)$$

$$p_0 + E_k - E_{pk} \simeq p_0 + \vec{p} \cdot \hat{k}, \quad (4.16)$$

$$p_0 - E_k + E_{pk} \simeq p_0 - \vec{p} \cdot \hat{k}, \quad (4.17)$$

and Eqs. (4.12) and (4.13) can be simplified as

$$T \sum_{k_0} \tilde{\Delta}(K) \Delta(P - K) \simeq \left[ \frac{n_F(E_k) + n_B(E_k)}{4kE_k} \left( \frac{1}{p_0 + \vec{p} \cdot \hat{k}} - \frac{1}{p_0 - \vec{p} \cdot \hat{k}} \right) \right], \quad (4.18)$$

$$T \sum_{k_0} k_0 \tilde{\Delta}(K) \Delta(P - K) \simeq - \left[ \frac{n_F(E_k) + n_B(E_k)}{4E_k} \left( \frac{1}{p_0 + \vec{p} \cdot \hat{k}} + \frac{1}{p_0 - \vec{p} \cdot \hat{k}} \right) \right]. \quad (4.19)$$

Using these, the tree level self-energy in (4.3) can be written as

$$\Sigma_1(P) = 2g^2 C_F \int \frac{k^2 dk}{4\pi^2} \left( \frac{n_F(E_k) + n_B(E_k)}{E_k} \right) \int \frac{d\Omega}{4\pi} \frac{\hat{K}}{P \cdot \hat{K}}, \quad (4.20)$$

where  $\hat{K} = (1, \hat{k})$ . Within HTL approximation ( $T \gg m_f$ ), one can neglect the mass term in  $E_k$  and replace it by  $k$ . Now, the momentum integral becomes a trivial one and eventually yields

$$\begin{aligned} \Sigma_1(P) &= 2g^2 C_F \int \frac{k dk}{4\pi^2} (n_F(k) + n_B(k)) \int \frac{d\Omega}{4\pi} \frac{\hat{K}}{P \cdot \hat{K}}, \\ &= \frac{g^2 T^2 C_F}{8} \int \frac{d\Omega}{4\pi} \frac{\hat{K}}{P \cdot \hat{K}}. \end{aligned} \quad (4.21)$$

We also note that for nonzero chemical potential, one can replace the Fermi distribution function  $n_F(k)$  as  $\frac{1}{2} [n_F(k + \mu) + n_F(k - \mu)]$  and instead of Eq. (4.21), one eventually obtains

$$\Sigma_1(P) = \frac{g^2 T^2 C_F}{8} (1 + 4\hat{\mu}^2) \int \frac{d\Omega}{4\pi} \frac{\hat{K}}{P \cdot \hat{K}}, \quad (4.22)$$

where  $\hat{\mu} = \frac{\mu}{2\pi T}$ .

## 4.2 Evaluation of $\Sigma_2(P)$

We now try to evaluate the leading order correction  $\mathcal{O}[q_f B]$  through the self energy,  $\Sigma_2(P)$ , due to weak external magnetic field as

$$\begin{aligned}\Sigma_2(P) &= g^2 C_F(q_f B) \int \frac{d^4 K}{(2\pi)^4} \gamma_\mu \gamma_1 \gamma_2 \frac{(\not{K}_\parallel + m_f)}{(K^2 - m_f^2)^2} \gamma^\mu \Delta(P - K) \\ &= g^2 C_F(q_f B) \int \frac{d^4 K}{(2\pi)^4} \gamma_\mu \gamma_1 \gamma_2 \not{K}_\parallel \gamma^\mu \tilde{\Delta}^2(K) \Delta(P - K).\end{aligned}\quad (4.23)$$

Here, we use the identity

$$i\gamma_1 \gamma_2 \not{K}_\parallel = \gamma_5 [(K \cdot b)\not{b} - (K \cdot u)\not{u}], \quad (4.24)$$

where  $u = (1, 0, 0, 0)$  is the four velocity of the heatbath in the rest frame and  $b = (0, 0, 0, 1)$  is introduced because of the anisotropy of the constant magnetic field along the  $z$  direction. This yields

$$\begin{aligned}\Sigma_2(P) &= ig^2 C_F \gamma_5(q_f B) \int \frac{d^4 K}{(2\pi)^4} \gamma_\mu [(K \cdot b)\not{b} - (K \cdot u)\not{u}] \gamma^\mu \tilde{\Delta}^2(K) \Delta(P - K) \\ &= -2ig^2 C_F \gamma_5(q_f B) \int \frac{d^4 K}{(2\pi)^4} [(K \cdot b)\not{b} - (K \cdot u)\not{u}] \tilde{\Delta}^2(K) \Delta(P - K) \\ &= 2g^2 C_F \gamma_5(q_f B) \sum_{\{K\}} [(K \cdot b)\not{b} - (K \cdot u)\not{u}] \tilde{\Delta}^2(K) \Delta(P - K).\end{aligned}\quad (4.25)$$

Now we have to evaluate the following sums:  $T \sum_{k_0} \tilde{\Delta}^2(K) \Delta(P - K)$  and  $T \sum_{k_0} k_0 \tilde{\Delta}^2(K) \Delta(P - K)$ . We can easily express these sums, respectively, in terms of the simpler ones evaluated earlier in subsec. 4.1 as

$$\begin{aligned}T \sum_{k_0} \tilde{\Delta}^2(K) \Delta(P - K) &= \frac{\partial}{\partial m_f^2} \left( T \sum_{k_0} \tilde{\Delta}(K) \Delta(P - K) \right) \\ &= \frac{\partial}{\partial m_f^2} \left[ \frac{n_F(E_k) + n_B(E_k)}{4kE_k} \left( \frac{1}{p_0 + \vec{p} \cdot \hat{k}} - \frac{1}{p_0 - \vec{p} \cdot \hat{k}} \right) \right],\end{aligned}\quad (4.26)$$

$$\begin{aligned}T \sum_{k_0} k_0 \tilde{\Delta}^2(K) \Delta(P - K) &= \frac{\partial}{\partial m_f^2} \left( T \sum_{k_0} k_0 \tilde{\Delta}(K) \Delta(P - K) \right) \\ &= -\frac{\partial}{\partial m_f^2} \left[ \frac{n_F(E_k) + n_B(E_k)}{4E_k} \left( \frac{1}{p_0 + \vec{p} \cdot \hat{k}} + \frac{1}{p_0 - \vec{p} \cdot \hat{k}} \right) \right].\end{aligned}\quad (4.27)$$

Using these, the self-energy in Eq. (4.25) can be written as

$$\begin{aligned}\Sigma_2(P) &= -2g^2 C_F \gamma_5(q_f B) \frac{\partial}{\partial m_f^2} \int \frac{k^2 dk}{4\pi^2} \left( \frac{n_F(E_k) + n_B(E_k)}{E_k} \right) \\ &\quad \times \int \frac{d\Omega}{4\pi} \frac{[(\hat{K} \cdot b)\not{b} - (\hat{K} \cdot u)\not{u}]}{P \cdot \hat{K}},\end{aligned}\quad (4.28)$$

where  $\hat{K} = (1, \hat{k})$ . Now, changing the variables as  $y = m_f/T$  and  $x = k/T$ , the momentum integral can be written as

$$\frac{\partial}{\partial m_f^2} \int \frac{k^2 dk}{4\pi^2} \left( \frac{n_F(E_k) + n_B(E_k)}{E_k} \right) = \frac{\partial}{\partial y^2} \int \frac{x^2 dx}{4\pi^2} \left( \frac{n_F(\sqrt{x^2 + y^2}) + n_B(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \right). \quad (4.29)$$

The integrals can be represented by the well-known functions [78, 79] as

$$h_{n+1}(y) = \frac{1}{\Gamma(n+1)} \int_0^\infty \frac{dx x^n}{\sqrt{x^2 + y^2}} n_B(\sqrt{x^2 + y^2}), \quad (4.30)$$

$$f_{n+1}(y) = \frac{1}{\Gamma(n+1)} \int_0^\infty \frac{dx x^n}{\sqrt{x^2 + y^2}} n_F(\sqrt{x^2 + y^2}), \quad (4.31)$$

which satisfy the following recursion relations

$$\frac{\partial h_{n+1}}{\partial y^2} = -\frac{h_{n-1}}{2n}, \quad (4.32)$$

$$\frac{\partial f_{n+1}}{\partial y^2} = -\frac{f_{n-1}}{2n}. \quad (4.33)$$

Therefore, in terms of these functions the leading order correction  $\mathcal{O}[q_f B]$  term of the quark self-energy in Eq. (4.28) can be written as

$$\Sigma_2(P) = 2g^2 C_F \gamma_5 (q_f B) \frac{[h_1(y) + f_1(y)]}{8\pi^2} \int \frac{d\Omega}{4\pi} \frac{[(\hat{K} \cdot b)\not{u} - (\hat{K} \cdot u)\not{b}]}{P \cdot \hat{K}}. \quad (4.34)$$

In the regime of HTL perturbation theory and weak magnetic field [78, 79] one can use the high temperature expansion of  $h_n$  and  $f_n$  to write down the expression for  $h_1$  and  $f_1$  as

$$h_1(y) = \frac{\pi}{2y} + \frac{1}{2} \ln\left(\frac{y}{4\pi}\right) + \frac{1}{2}\gamma_E + \dots, \quad (4.35)$$

$$f_1(y) = -\frac{1}{2} \ln\left(\frac{y}{\pi}\right) - \frac{1}{2}\gamma_E + \dots. \quad (4.36)$$

In case of finite chemical potential the expression for  $f_1$  gets modified as

$$f_1(y) = -\frac{1}{2} \ln\left(\frac{y}{4\pi}\right) + \frac{1}{4}\aleph(z) + \dots, \quad (4.37)$$

where the function  $\aleph(z)$  is defined in Eq. (B.13) in Appendix B. Now, the leading order correction  $\mathcal{O}[q_f B]$  term of the quark self-energy in Eq. (4.34) in presence of a weak magnetic field becomes

$$\Sigma_2(P) = -4g^2 C_F \gamma_5 M_{B,f}^2 \int \frac{d\Omega}{4\pi} \frac{[(\hat{K} \cdot b)\not{u} - (\hat{K} \cdot u)\not{b}]}{P \cdot \hat{K}}, \quad (4.38)$$

where the magnetic mass <sup>1</sup> for a given flavour  $f$  is given as

$$M_{B,f}^2 = \frac{q_f B}{16\pi^2} \left[ -\frac{1}{4}\aleph(z) - \frac{\pi T}{2m_f} - \frac{\gamma_E}{2} \right] \quad (4.39)$$

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<sup>1</sup>For zero chemical potential it becomes [69]  $M_{B,f}^2 = \frac{q_f B}{16\pi^2} \left[ \ln 2 - \frac{\pi T}{2m_f} \right]$ .

Finally, combining Eqs. (4.21) and (4.38) with Eq. (4.1) the total quark self-energy contribution of  $\mathcal{O}[q_f B]$  in presence of a weak magnetic field within HTL approximation can be written as

$$\begin{aligned} \Sigma(P) = & \frac{g^2 C_F T^2}{8} (1 + 4\hat{\mu}^2) \int \frac{d\Omega}{4\pi} \frac{\hat{K}}{P \cdot \hat{K}} \\ & - 4g^2 C_F \gamma_5 M_{B,f}^2 \int \frac{d\Omega}{4\pi} \frac{[(\hat{K} \cdot b)\not{b} - (\hat{K} \cdot u)\not{b}]}{P \cdot \hat{K}}. \end{aligned} \quad (4.40)$$

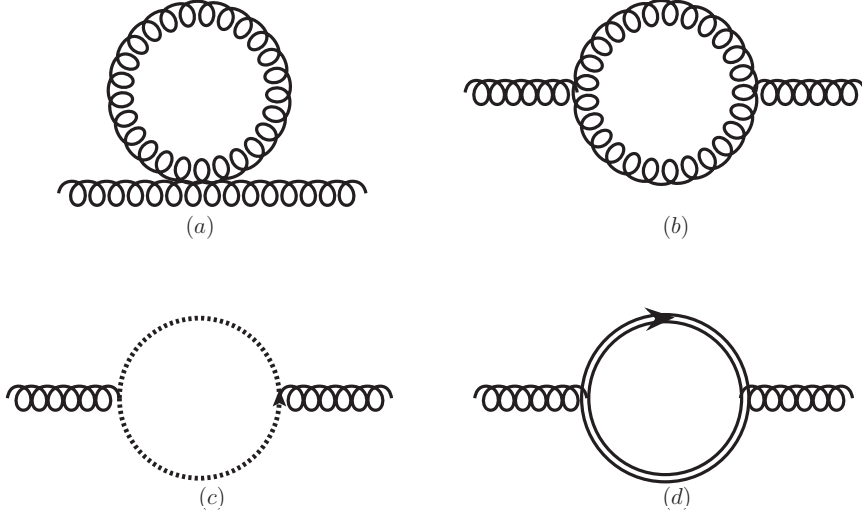
## 5 QCD Debye mass in presence of an external magnetic field

Both the transverse and the longitudinal gluon self-energy functions, which are essential for evaluating the gluonic part of the thermodynamic free energy, can be expressed in terms of the Debye screening mass  $m_D$  as

$$\Pi_T(Q) = \frac{m_D^2}{(d-1)n_q^2} [\tau^{00}(Q, -Q) - 1 + n_q^2], \quad (5.1)$$

$$\Pi_L(Q) = m_D^2 [1 - \tau^{00}(Q, -Q)], \quad (5.2)$$

where the symmetric tensor  $\tau^{\mu\nu}$  and the four vector  $n_q^\mu$  are defined in Appendix A.



**Figure 2.** Gluon self-energy diagrams in weak magnetic field approximation. (a) Four gluon vertex, (b) Three gluon vertex, (c) Ghost loop and (d) Fermion loop. The double line in the fermionic loop indicates the modified fermionic propagators in presence of weak magnetic field.

This in turn suggests us to study the modification of the Debye screening mass  $m_D$  in presence of a weak external magnetic field. The Debye mass in QCD is related with the temporal component of the gluon polarization tensor ( $\Pi_{\mu\nu}$ ) via

$$m_D^2 = -\Pi_{00}(q_0 = 0, q \rightarrow 0), \quad (5.3)$$

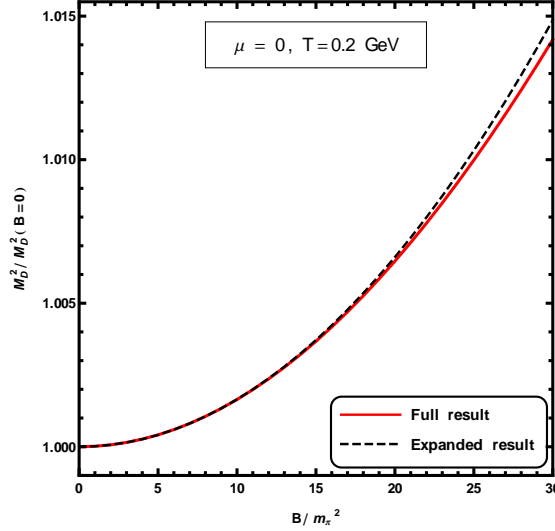
where  $Q \equiv (q_0, q)$  is the external gluon four momentum and  $(q_0 = 0, q \rightarrow 0)$  is called the static limit.

The gluon polarization tensor or the gluon self-energy can be evaluated from the four contributing diagrams as shown in Fig. 2 and eventually in the absence of external magnetic field the Debye mass comes out to be [60]

$$\hat{m}_D^2(B=0) = \frac{g^2}{12\pi^2} \left( \left( N_c + \frac{N_f}{2} \right) + 6N_f \hat{\mu}^2 \right), \quad (5.4)$$

where  $\hat{m}_D = m_D/2\pi T$ . Now in presence of magnetic field only the fermionic diagram in Fig. 2(d) will pick up the correction because of the modified fermionic propagators. The electromagnetic Debye mass in presence of magnetic field was computed in [14, 80]. Generalizing this to QCD we obtain the expression for the modified QCD Debye mass for QCD at finite chemical potential and an arbitrary magnetic field as

$$\begin{aligned} \hat{m}_D^2 = & \frac{g^2 N_c}{12\pi^2} + \sum_f \frac{g^2 q_f B}{8\pi^4 T^2} \int_0^\infty e^{-x} dx \\ & \times \sum_{l=1}^\infty (-1)^{l+1} \cosh(2l\pi\hat{\mu}) \coth\left(\frac{q_f B l^2}{4xT^2}\right) \exp\left(-\frac{m_f^2 l^2}{4xT^2}\right). \end{aligned} \quad (5.5)$$



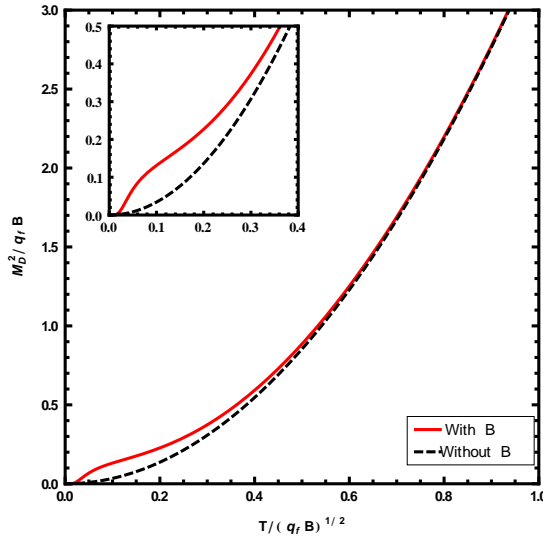
**Figure 3.** Comparison of the scaled one-loop Debye masses in Eqs.(5.5) and (5.6) varying with scaled magnetic field for  $N_f = 2$  and  $\mu = 0$ .

In the weak field approximation ( $T^2 > q_f B$ ), the square of Debye mass can be obtained from Eq. (5.5) by expanding  $\coth(q_f B l^2 / 4xT^2)$  as

$$\hat{m}_D^2 \simeq \frac{g^2}{12\pi^2} \left( \left( N_c + \frac{N_f}{2} \right) + 6N_f \hat{\mu}^2 \right)$$

$$+ \sum_f \frac{g^2(q_f B)^2}{48\pi^4 T^4} \sum_{l=1}^{\infty} (-1)^{l+1} l^2 \cosh(2l\pi\hat{\mu}) K_0\left(\frac{m_f l}{T}\right) + \mathcal{O}[(q_f B)^4], \quad (5.6)$$

where  $K_n(z)$  represents the modified Bessel function of the second kind. In Eq. (5.6) the first term is the Debye mass contribution in the absence of the external magnetic field whereas the second term is the correction due to the presence of the weak external magnetic field. In Fig. 3 the full expression in Eq. (5.5) and the weak field expression in Eq. (5.6) are displayed as a scaled magnetic field. In the strong field limit ( $B/m_\pi^2 \geq 20$ ) the weak field result deviates slightly from that of the full result. However, there is no difference between the two in limit  $B/m_\pi^2 < 20$ , so it is a good approximation to work with Eq. (5.6) in the weak field limit.



**Figure 4.** Comparison of the scaled one-loop Debye masses varying with the scaled temperature for  $N_f = 2$  and  $\mu = 0$ .

In Fig. 4 we explicitly show some interesting features in the Debye mass due to the inclusion of the arbitrary external magnetic field by comparing it with the non-magnetized case. The  $x$ -axis is chosen to be  $T/(q_f B)^{1/2}$ , so that one can understand different limits of the magnetic fields. We note that Fig. 4 clearly reveals three different scales,  $m_f$ ,  $T$  and  $B$ , associated with the hot and dense magnetized medium. At  $T = 0$ , the  $m_D = 0$  and it gradually increases with  $T$  as long as  $T < m_f < (q_f B)^{1/2}$ . When  $T \sim m_f < (q_f B)^{1/2}$  a shoulder appears in  $m_D$  and then it increases a bit slowly<sup>2</sup> in the regime  $m_f \leq T \leq (q_f B)^{1/2}$ . This strong field behaviour of  $m_D$  is shown in the inset. Now we also note that if the thermal scale is higher than the magnetic scale ( $T \gg (q_f B)^{1/2}$ ), then  $m_D$  increases with  $T$  like the usual hot but unmagnetized medium as shown in Fig. 4.

<sup>2</sup>We further note that the shoulder is pushed towards the higher  $T$  as quark mass increases[14]. The appearance of the shoulder depends on the strength of two scales, viz.,  $m_f$  and  $T$  associated with the hot magnetized system [14].

## 6 HTL Free Energy and Pressure in a weak magnetic field

### 6.1 Quark part

The quark part of the one-loop HTL thermodynamic free energy can be written from Eq. (2.5) and Eq. (2.6) as

$$F_q^{HTL} = -N_c N_f \int \frac{d^4 P}{(2\pi)^4} \ln (\det [\not{P} - \Sigma(P)]). \quad (6.1)$$

We note that in HTL approximation one can neglect the current quark mass with respect to the thermal mass appearing in the expression of the self-energy  $\Sigma(P)$ . As evaluated in section 4, we straightway write down the expression for the quark self-energy within HTL approximation in weak magnetic field limit from Eq. (4.40) as

$$\begin{aligned} \Sigma(P) &= \frac{g^2 T^2 C_F}{8} (1 + 4\hat{\mu}^2) \int \frac{d\Omega}{4\pi} \frac{\hat{K}}{P \cdot \hat{K}} + 4\gamma_5 g^2 C_F M_{B,f}^2 \int \frac{d\Omega}{4\pi} \frac{[(\hat{K} \cdot u)\not{b} - (\hat{K} \cdot b)\not{u}]}{P \cdot \hat{K}}, \\ &= m_q^2 \int \frac{d\Omega}{4\pi} \frac{(\gamma_0 - \gamma_i \hat{k}_i)}{p_0 - \vec{p} \cdot \hat{k}} + m_{\text{eff}}^2 \gamma_5 \int \frac{d\Omega}{4\pi} \frac{(\gamma_0 \hat{k}_3 - \gamma_3)}{p_0 - \vec{p} \cdot \hat{k}}, \\ &= m_q^2 \left[ \gamma_0 \int \frac{d\Omega}{4\pi} \frac{1}{p_0 - \vec{p} \cdot \hat{k}} - \gamma_i \int \frac{d\Omega}{4\pi} \frac{\hat{k}_i}{p_0 - \vec{p} \cdot \hat{k}} \right] \\ &\quad + m_{\text{eff}}^2 \gamma_5 \left[ \gamma_0 \int \frac{d\Omega}{4\pi} \frac{\hat{k}_3}{p_0 - \vec{p} \cdot \hat{k}} - \gamma_3 \int \frac{d\Omega}{4\pi} \frac{1}{p_0 - \vec{p} \cdot \hat{k}} \right], \\ &= -a(p_0, p) \gamma_5 \gamma_0 - b(p_0, p) \gamma_5 \gamma_3 + c_1(p_0, p) p_0 \gamma_0 + d_1(p_0, p) p_i \gamma_i, \end{aligned} \quad (6.2)$$

where  $i$  runs from 1 to 3 and

$$m_{\text{eff}}^2 = 4g^2 C_F M_B^2; \quad M_B^2 = \sum_f M_{B,f}^2, \quad (6.3)$$

$$m_q^2 = \frac{g^2 C_F T^2}{8} (1 + 4\hat{\mu}^2), \quad (6.4)$$

$$a(p_0, p) = -m_{\text{eff}}^2 \int \frac{d\Omega}{4\pi} \frac{\hat{k}_3}{p_0 - \vec{p} \cdot \hat{k}} = m_{\text{eff}}^2 \frac{1}{p} [1 - \mathcal{T}_P], \quad (6.5)$$

$$b(p_0, p) = m_{\text{eff}}^2 \int \frac{d\Omega}{4\pi} \frac{1}{p_0 - \vec{p} \cdot \hat{k}} = m_{\text{eff}}^2 \frac{1}{p_0} \mathcal{T}_P, \quad (6.6)$$

$$c_1(p_0, p) = \frac{m_q^2}{p_0} \int \frac{d\Omega}{4\pi} \frac{1}{p_0 - \vec{p} \cdot \hat{k}} = m_q^2 \frac{1}{p_0^2} \mathcal{T}_P, \quad (6.7)$$

$$d_1(p_0, p) = -\frac{m_q^2}{p^2} \int \frac{d\Omega}{4\pi} \frac{\vec{p} \cdot \hat{k}}{p_0 - \vec{p} \cdot \hat{k}} = m_q^2 \frac{1}{p^2} [1 - \mathcal{T}_P], \quad (6.8)$$

with

$$\mathcal{T}_P = \int \frac{d\Omega}{4\pi} \frac{p_0}{p_0 - \vec{p} \cdot \hat{k}}, \quad (6.9)$$

which is an integral defined as the angular average over the cosine of  $\angle \vec{p}, \hat{k}$ . We also note that  $\vec{p}$  is taken to be along the  $z$  direction in Eq. (6.5). Using Eq. (6.2) one can now write

$$\not{P} - \Sigma(P) = [c(p_0, p)p_0\gamma_0 - d(p_0, p)p_i\gamma_i + a(p_0, p)\gamma_5\gamma_0 + b(p_0, p)\gamma_5\gamma_3], \quad (6.10)$$

where

$$c(p_0, p) = 1 - c_1(p_0, p), \quad (6.11)$$

$$d(p_0, p) = 1 - d_1(p_0, p). \quad (6.12)$$

In terms of these introduced notations, we evaluate the determinant of Eq. (6.10) as

$$\begin{aligned} \det [\not{P} - \Sigma(P)] &= (c^2 p_0^2 - d^2 p^2 + a^2 - b^2)^2 - 4(p_0 a c + p_3 b d)^2 \\ &= A_0^2 - A_s^2. \end{aligned} \quad (6.13)$$

Combining Eqs.(6.1) and (6.13), the one-loop quark HTL free energy in presence of a weak magnetic field can be written as

$$\begin{aligned} F_q^{HTL} &= -N_c N_f \int \frac{d^4 P}{(2\pi)^4} \ln(A_0^2 - A_s^2) \\ &= -2N_c N_f \int \frac{d^4 P}{(2\pi)^4} \ln(P^2) - N_c N_f \int \frac{d^4 P}{(2\pi)^4} \ln\left(\frac{A_0^2 - A_s^2}{P^4}\right) \\ &= N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left(1 + \frac{120}{7}\hat{\mu}^2 + \frac{240}{7}\hat{\mu}^4\right) \right. \\ &\quad \left. - \int \frac{d^4 P}{(2\pi)^4} \ln\left[\frac{(A_0 + A_s)(A_0 - A_s)}{P^4}\right] \right], \end{aligned} \quad (6.14)$$

where  $N_c$  is the number of colour. Now the argument of the logarithm in Eq. (6.14) can be simplified using Eq. (6.13) as

$$\begin{aligned} \frac{(A_0 + A_s)(A_0 - A_s)}{P^4} &= 1 + 2 \left( \frac{c_1(c_1 - 2)p_0^2 - d_1(d_1 + 2)p^2 + a^2 - b^2}{P^2} \right) \\ &\quad + \frac{(c_1(c_1 - 2)p_0^2 - d_1(d_1 + 2)p^2 + a^2 - b^2)^2 - 4(acp_0 + bdp_3)^2}{P^4}. \end{aligned} \quad (6.15)$$

In high temperature limit, the logarithmic term in Eq. (6.14) can be expanded in series of coupling constant  $g$  keeping terms up to  $\mathcal{O}(g^4)$  as

$$\begin{aligned} \ln\left[\frac{(A_0 + A_s)(A_0 - A_s)}{P^4}\right] &= 2 \left( \frac{c_1^2 p_0^2 - d_1^2 p^2 + a^2 - b^2 - 2c_1 p_0^2 - 2d_1 p^2}{P^2} \right) \\ &\quad - 4 \left( \frac{(c_1 p_0^2 + d_1 p^2)^2 + (ap_0 + bp_3)^2}{P^4} \right) + \mathcal{O}(g^6), \end{aligned} \quad (6.16)$$

with

$$(a^2 - b^2) = m_{\text{eff}}^4 \left[ \frac{1}{p^2} + \frac{\mathcal{T}_P^2}{p^2} - \frac{\mathcal{T}_P^2}{p_0^2} - \frac{2\mathcal{T}_P}{p^2} \right], \quad (6.17)$$



$$(ap_0 + bp_3)^2 = m_{\text{eff}}^4 \left[ \frac{p_0^2}{p^2} (1 + \mathcal{T}_P^2 - 2\mathcal{T}_P) + \frac{\mathcal{T}_P^2}{p_0^2} p_3^2 \right], \quad (6.18)$$

$$(c_1 p_0^2 + d_1 p^2) = m_q^2, \quad (6.19)$$

$$(c_1^2 p_0^2 - d_1^2 p^2) = m_q^4 \left[ \frac{\mathcal{T}_P^2}{p_0^2} - \frac{(1 - \mathcal{T}_P)^2}{p^2} \right]. \quad (6.20)$$

So, upto  $\mathcal{O}(g^4)$  the one-loopfree energy can be written as,

$$\begin{aligned} F_q^{HTL} &= N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) + 4m_q^2 \int \frac{d^4 P}{(2\pi)^4} \frac{1}{P^2} \right. \\ &\quad - m_q^4 \int \frac{d^4 P}{(2\pi)^4} \left[ \frac{2\mathcal{T}_P^2}{p_0^2 P^2} - \frac{4}{P^4} - \frac{2}{p^2 P^2} - \frac{2\mathcal{T}_P^2}{p^2 P^2} + \frac{4\mathcal{T}_P}{p^2 P^2} \right] \\ &\quad \left. - \frac{m_{\text{eff}}^4}{N_f} \int \frac{d^4 P}{(2\pi)^4} \left[ \frac{2}{P^2} \left( \frac{1}{p^2} + \frac{\mathcal{T}_P^2}{p^2} - \frac{\mathcal{T}_P^2}{p_0^2} - \frac{2\mathcal{T}_P}{p^2} \right) - \frac{4}{P^4} \left( \frac{p_0^2}{p^2} (1 + \mathcal{T}_P^2 - 2\mathcal{T}_P) + \frac{\mathcal{T}_P^2}{p_0^2} p_3^2 \right) \right] \right], \\ &= N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) + 4m_q^2 \sum_{\{P\}} \frac{1}{P^2} \right. \\ &\quad - m_q^4 \left( \sum_{\{P\}} \frac{2\mathcal{T}_P^2}{p_0^2 P^2} - \sum_{\{P\}} \frac{4}{P^4} - \sum_{\{P\}} \frac{2}{p^2 P^2} - \sum_{\{P\}} \frac{2\mathcal{T}_P^2}{p^2 P^2} + \sum_{\{P\}} \frac{4\mathcal{T}_P}{p^2 P^2} \right) \\ &\quad + \frac{2m_{\text{eff}}^4}{N_f} \left( \sum_{\{P\}} \frac{1}{p^2 P^2} + \sum_{\{P\}} \frac{\mathcal{T}_P^2}{p^2 P^2} + \sum_{\{P\}} \frac{\mathcal{T}_P^2}{p_0^2 P^2} - \sum_{\{P\}} \frac{2\mathcal{T}_P}{p^2 P^2} \right) \\ &\quad \left. + \frac{4m_{\text{eff}}^4}{N_f} \left( \sum_{\{P\}} \frac{1}{P^4} + \sum_{\{P\}} \frac{\mathcal{T}_P^2}{P^4} - 2 \sum_{\{P\}} \frac{\mathcal{T}_P}{P^4} + \sum_{\{P\}} \frac{\mathcal{T}_P^2}{p_0^2 P^4} p_3^2 \right) \right], \\ &= N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) \right. \\ &\quad + \frac{m_q^2 T^2}{6} (1 + 12\hat{\mu}^2) + 4m_q^4 \left[ 1 + \frac{1 - 2A_1 + A_2 - A_3}{2 - d} \right] \sum_{\{P\}} \frac{1}{P^4} \\ &\quad \left. + \frac{4m_{\text{eff}}^4}{N_f} \left[ -1 + A_4 + \frac{1 - 2A_1 + A_2 + A_3 + 2A_4 + 2A_5 + A_6}{2 - d} - \frac{2}{d - 5} + \frac{A_4 + A_6}{d} \right] \sum_{\{P\}} \frac{1}{P^4} \right], \quad (6.21) \end{aligned}$$

where all the sum-integrals are provided in the Appendix B.  $A_i$ 's are the c-integrations arising due to angular integral  $\mathcal{T}_P$  and evaluated in the Appendix B. Using the expressions for various sum-integrals obtained in the Appendix B we can write

$$F_q^{HTL} = N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) \right.$$

$$\begin{aligned}
& + \frac{m_q^2 T^2}{6} (1 + 12\hat{\mu}^2) + 4m_q^4 \left[ \left( \frac{\pi^2}{3} - 2 \right) \epsilon \right] \sum_{\{P\}} \frac{1}{P^4} \\
& + \frac{m_{\text{eff}}^4}{N_f} \left[ \frac{4}{9} (2\pi^2 - 15) - \frac{2}{27} (-72\zeta(3) + 7\pi^2 + 60) \epsilon \right] \sum_{\{P\}} \frac{1}{P^4} \Big], \\
& = N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) + \frac{m_q^2 T^2}{6} (1 + 12\hat{\mu}^2) + \frac{m_q^4}{12\pi^2} (\pi^2 - 6) \right. \\
& + \frac{m_{\text{eff}}^4}{36\pi^2 N_f} \left[ \frac{2\pi^2 - 15}{\epsilon} + \left( 2 \ln \frac{\Lambda}{4\pi T} - \mathfrak{N}(z) \right) (2\pi^2 - 15) + 12\zeta(3) - \frac{7\pi^2}{6} - 10 \right] \Big], \\
& = N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) + \frac{g^2 C_F T^4}{48} (1 + 4\hat{\mu}^2) (1 + 12\hat{\mu}^2) \right. \\
& + \frac{g^4 C_F^2 T^4}{768\pi^2} (1 + 4\hat{\mu}^2)^2 (\pi^2 - 6) + \frac{4N_c g^4 C_F^2}{9\pi^2} M_B^4 \\
& \times \left[ \left( 2 \ln \frac{\Lambda}{4\pi T} - \mathfrak{N}(z) \right) (2\pi^2 - 15) + 12\zeta(3) - \frac{7\pi^2}{6} - 10 \right] \Big] \\
& + \frac{4N_c g^4 C_F^2}{9\epsilon} M_B^4 \left( 2 - \frac{15}{\pi^2} \right). \tag{6.22}
\end{aligned}$$

Now, the divergence that depends on the external magnetic field  $B$  can be removed [31] by redefining the magnetic field  $B$  in the tree-level free energy as

$$F_0 = B^2 \rightarrow B^2 \left[ 1 - \frac{8N_c g^4 C_F^2}{9\epsilon} \frac{M_B^4}{B^2} \left( 2 - \frac{15}{\pi^2} \right) \right]. \tag{6.23}$$

## 6.2 Gluonic part

The modification of the gluonic part of the one-loop HTL free energy solely depends on the modification of the Debye mass in presence of a weak external magnetic field as discussed in section 5. The explicit evaluation of the gluonic part of the one-loop HTL free energy for hot and dense QCD matter in absence of magnetic field in terms of the Debye mass is done in [58, 60]. Here we just write down the gluon free energy with the Debye mass now modified in presence of weak magnetic field via Eq. (5.6) as

$$\begin{aligned}
F_g^{HTL} = & -\frac{d_A \pi^2 T^4}{45} \left[ 1 - \frac{15}{2} \hat{m}_D^2 + 30 \hat{m}_D^3 \right. \\
& \left. + \frac{45}{8} \left( \frac{1}{\epsilon} + 2 \ln \frac{\Lambda}{4\pi T} - 7 + 2\gamma_E + \frac{2\pi^2}{3} \right) \hat{m}_D^4 + \mathcal{O}(g^6) \right]. \tag{6.24}
\end{aligned}$$

The divergence in Eq. (6.24) is regulated through the HTL counter term as given in Eq. (2.4).

### 6.3 Total thermodynamic free energy

Now, the total one-loop HTL renormalized free energy upto  $\mathcal{O}(g^4)$  reads from Eq. (2.3) by using Eqs.(2.4), (6.22), (6.23) and (6.24) as

$$\begin{aligned}
F^{HTL}(T, \mu, B, \Lambda) &= F_q^{HTL}(T, \mu, B, \Lambda) + F_0 + F_g^{HTL}(T, \mu, B, \Lambda) + \Delta\mathcal{E}_0 \\
&= N_c N_f \left[ -\frac{7\pi^2 T^4}{180} \left( 1 + \frac{120\hat{\mu}^2}{7} + \frac{240\hat{\mu}^4}{7} \right) + \frac{g^2 C_F T^4}{48} (1 + 4\hat{\mu}^2) (1 + 12\hat{\mu}^2) \right. \\
&\quad + \frac{g^4 C_F^2 T^4}{768\pi^2} (1 + 4\hat{\mu}^2)^2 (\pi^2 - 6) + \frac{4N_c g^4 C_F^2}{9\pi^2} M_B^4 \\
&\quad \times \left[ \left( 2\ln \frac{\Lambda}{4\pi T} - \Re(z) \right) (2\pi^2 - 15) + 12\zeta(3) - \frac{7\pi^2}{6} - 10 \right] + \frac{1}{2} B^2 \\
&\quad \left. - d_A \frac{\pi^2 T^4}{45} \left[ 1 - \frac{15}{2} \hat{m}_D^2 + 30\hat{m}_D^3 + \frac{45}{8} \left( 2\ln \frac{\Lambda}{4\pi T} - 7 + 2\gamma_E + \frac{2\pi^2}{3} \right) \hat{m}_D^4 \right] \right]. \quad (6.25)
\end{aligned}$$

### 6.4 Pressure

The expression for the pressure of hot and dense QCD matter in one-loop HTLpt in presence of a weak magnetic field can now be written directly from the one-loop free energy as

$$P(T, \mu, B, \Lambda) = -F^{HTL}(T, \mu, B, \Lambda), \quad (6.26)$$

whereas the ideal gas limit of the pressure reads as

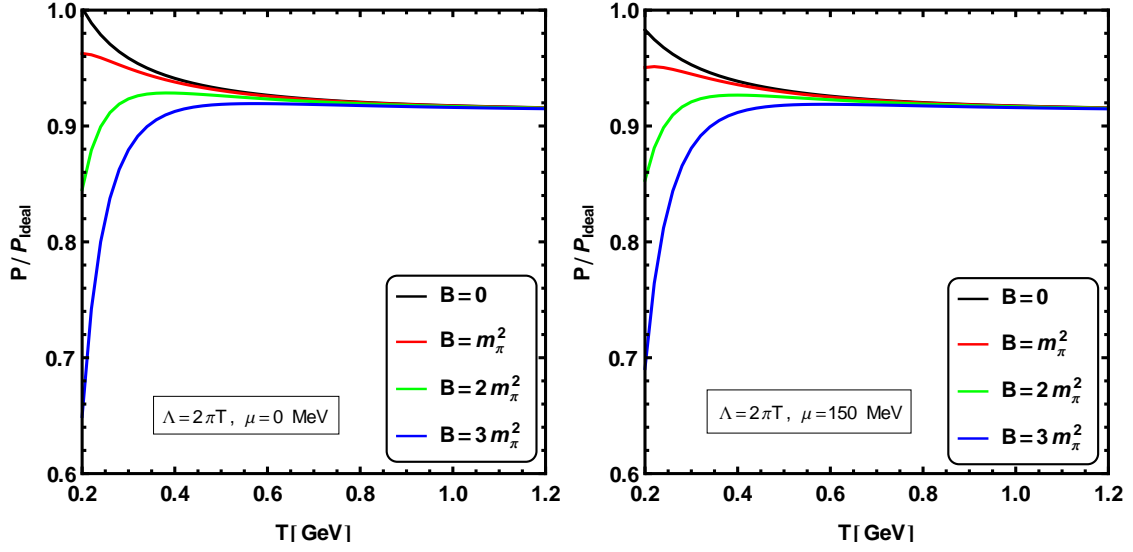
$$P_{\text{Ideal}}(T, \mu) = N_c N_f \frac{7\pi^2 T^4}{180} \left( 1 + \frac{120}{7} \hat{\mu}^2 + \frac{240}{7} \hat{\mu}^4 \right) + (N_c^2 - 1) \frac{\pi^2 T^4}{45}. \quad (6.27)$$

Also we use the one-loop running coupling

$$\alpha_s(\Lambda) = \frac{12\pi}{11N_c - 2N_f} \frac{1}{\ln \left( \Lambda^2 / \Lambda_{\overline{\text{MS}}}^2 \right)}, \quad (6.28)$$

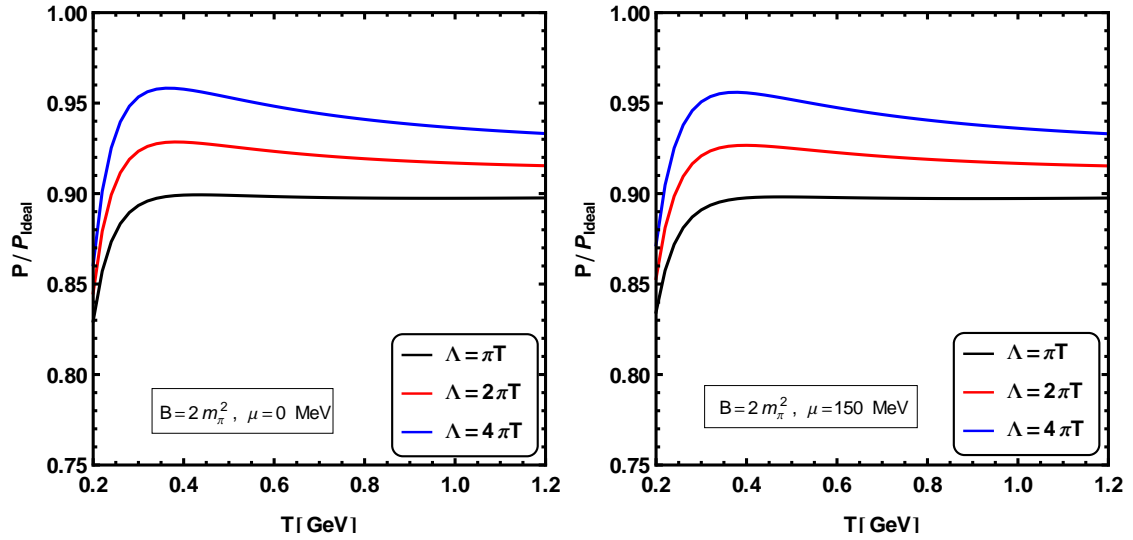
with  $\Lambda_{\overline{\text{MS}}} = 176$  MeV [81] and the renormalisation scale is chosen at its central value,  $\Lambda = 2\pi T$ .

In Fig. 5 we show the temperature variation of the one-loop HTLpt pressure in presence of weak magnetic fields of different strengths scaled with the QGP pressure at the ideal gas limit, both with zero and finite chemical potential. As seen the low  $T$  behaviour of the pressure is strongly affected by the presence of magnetic field whereas at high  $T$  it is obviously almost unaffected by the weak magnetic magnetic field. Nevertheless, we also note a specific difficulty with HTLpt which has to do with the fact that the one-loop HTLpt introduces an over-counting of some contributions [50–57]. This is because the loop expansion and the coupling expansion are not symmetrical in HTLpt. So, at each loop order in HTLpt the result is an infinite series in  $g$ . At leading order in HTLpt one only gets the correct perturbative coefficients for  $g^0$  and  $g^3$  when one expands in power of  $g$ . Thus, for a given order in  $g$  higher loop orders contribute and this is only corrected by making the calculation in higher loop-orders [58–68].



**Figure 5.** Variation of the scaled one-loop HTLpt pressure with temperature for  $N_f = 2$  with  $\mu = 0$  (left) and  $\mu = 150$  MeV (right) in presence of weak magnetic fields of various strengths.

To check the sensitivity with the renormalization scale  $\Lambda$  in Fig. 6 we have displayed the temperature variation of the scaled one-loop HTLpt pressure in presence of a constant weak magnetic field by varying  $\Lambda$  by a factor of two around its central value  $2\pi T$ , namely  $\Lambda = \pi T$ ,  $2\pi T$  and  $4\pi T$ , for both zero and finite chemical potential. It is found to depend moderately on the renormalization scale  $\Lambda$ .



**Figure 6.** Variation of the scaled one-loop HTLpt pressure with temperature for  $N_f = 2$  with  $\mu = 0$  (left) and  $\mu = 150$  MeV (right) in presence of a weak magnetic field of strength  $m_\pi^2$  for different values of renormalization scale  $\Lambda$ .

## 7 Conclusion

In this paper we presented a systematic framework to evaluate the QCD pressure within one-loop HTLpt in presence of a weak external magnetic field. The total pressure is the sum of three contributions coming from (a) the fermionic/quark part, (b) the gluonic part and (c) the tree level free energy due to the constant magnetic field. We note that the inclusion of an external magnetic field largely affects the fermionic part. The modification of fermionic HTL propagator and hence the HTL quark self-energy in presence of the weak magnetic field are computed in details which eventually contributes into the fermionic part of the free energy. The divergence appeared therein is taken care of by redefining the magnetic field in the tree level free energy term. The gluonic part is largely unaffected except for the Debye screening mass that gets modified by the introduction of the external magnetic field. We have also discussed in details the modification of QCD Debye mass which depends on three scales, *viz.*, the quark mass, temperature and the magnetic field. By incorporating the changes in Debye mass we get the modified gluonic part of the free energy. The diverging part in the gluon free energy is regulated through the HTL counter term. We have also outlined a general drawback with one-loop HTLpt that introduces an over-counting of some contributions, as a remedy of which one needs to push the calculation to higher loop-order.

## 8 Acknowledgement

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## A Gluon self-energy

The gluon self-energy tensor in HTL approximation is given by

$$\Pi^{\mu\nu}(Q) = m_D^2 [\tau^{\mu\nu}(Q, -Q) - n^\mu n^\nu], \quad (\text{A.1})$$

where  $Q(q_0, q)$  is the external gluon four momentum and

$$\tau^{\mu\nu}(Q, -Q) = \left\langle x^\mu x^\nu \frac{Q \cdot n}{Q \cdot x} \right\rangle_{\hat{x}}, \quad (\text{A.2})$$

is symmetric in  $\mu$  and  $\nu$  and  $x(1, \hat{x})$  is the light like four vector defined in section 2, over whose spatial directions the angular average is taken. So the self-energy tensor eventually satisfy the following equations

$$Q_\mu \Pi^{\mu\nu}(Q) = 0, \quad (\text{A.3})$$

$$g_{\mu\nu} \Pi^{\mu\nu}(Q) = -m_D^2. \quad (\text{A.4})$$

The gluon self-energy can be expressed in terms of  $\Pi_T$  and  $\Pi_L$ , the transverse and longitudinal self-energy functions respectively, given by

$$\Pi_T(Q) = \frac{1}{d-1} (\delta^{ij} - \hat{q}^i \hat{q}^j) \Pi^{ij}(Q) \quad (\text{A.5})$$

$$\Pi_L(Q) = -\Pi^{00} \quad (\text{A.6})$$

In terms of these functions, the gluon self-energy tensor is given by

$$\Pi^{\mu\nu}(Q) = -\Pi_T(Q)T_q^{\mu\nu} - \frac{1}{n_q^2}\Pi_L(Q)L_q^{\mu\nu}, \quad (\text{A.7})$$

where

$$T_q^{\mu\nu} = g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} - \frac{n_q^\mu n_q^\nu}{n_q^2}, \quad (\text{A.8})$$

$$L_q^{\mu\nu} = \frac{n_q^\mu n_q^\nu}{n_q^2}, \quad (\text{A.9})$$

$$n_q^\mu = n^\mu - \frac{n \cdot Q}{Q^2} Q^\mu. \quad (\text{A.10})$$

The four vector  $n_q^\mu$  also satisfies  $Q \cdot n_q = 0$  and  $n_q^2 = 1 - \frac{(n \cdot Q)^2}{Q^2}$ . Using these now Eq. (A.4) can be reduced to the identity

$$(d-1)\Pi_T(Q) + \frac{1}{n_q^2}\Pi_L(Q) = m_D^2. \quad (\text{A.11})$$

So, now using Eq. (A.1) and Eq. (A.11) finally we can express both the transverse and the longitudinal self-energy functions in terms of the Debye screening mass  $m_D$  as

$$\Pi_T(Q) = \frac{m_D^2}{(d-1)n_q^2} [\tau^{00}(Q, -Q) - 1 + n_q^2], \quad (\text{A.12})$$

$$\Pi_L(Q) = m_D^2 [1 - \tau^{00}(Q, -Q)]. \quad (\text{A.13})$$

## B Sum-Integrals

The dimensionally regularized sum integrals are defined as,

$$\oint_{\{P\}} = \left( \frac{e^{\gamma_E} \Lambda^2}{4\pi} \right)^\epsilon T \sum_{\substack{p_0 = i\omega_n \\ \omega_n = (2n+1)\pi T - i\mu}} \int \frac{d^{d-2\epsilon} p}{(2\pi)^{d-2\epsilon}}, \quad (\text{B.1})$$

where  $\Lambda$  can be identified as the  $\bar{M}\bar{S}$  renormalization scale which also introduces the factor  $\left(\frac{e^{\gamma_E}}{4\pi}\right)^\epsilon$  along with it, with  $\gamma_E$  being the Euler-Mascheroni constant. Before listing all the sum-integrals used in our purpose, we note that they are inter related among themselves via

$$\oint_{\{P\}} \frac{1}{P^4} = -\frac{d-2}{2} \oint_{\{P\}} \frac{1}{p^2 P^2} = \frac{d-5}{d-4} \oint_{\{P\}} \frac{\mathcal{T}_P}{P^4}. \quad (\text{B.2})$$

### B.1 Simple one-loopsum-integrals

The list of fermionic sun-integrals needed are

1.

$$\sum_{\{P\}} \frac{1}{P^2} = \frac{T^2}{24} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} [1 + 12\hat{\mu}^2 + 2\epsilon (1 + 12\hat{\mu}^2 + 12\aleph(1, z))], \quad (\text{B.3})$$

2.

$$\sum_{\{P\}} \frac{1}{P^4} = \frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} - \aleph(z) \right], \quad (\text{B.4})$$

3.

$$\sum_{\{P\}} \frac{p^2}{P^6} = -\frac{3}{4} \frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} - \frac{2}{3} - \aleph(z) \right], \quad (\text{B.5})$$

4.

$$\sum_{\{P\}} \frac{1}{p^2 P^2} = -\frac{1}{(4\pi)^2} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} 2 \left[ \frac{1}{\epsilon} + 2 - \aleph(z) \right], \quad (\text{B.6})$$

5.

$$\sum_{\{P\}} \frac{p_3^2}{p^2 P^4} = \frac{1}{(4\pi)^2} \frac{1}{3} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + \frac{2}{3} - \aleph(z) \right], \quad (\text{B.7})$$

6.

$$\sum_{\{P\}} \frac{p_3^2}{p^4 P^2} = -\frac{1}{(4\pi)^2} \frac{2}{3} \left( \frac{\Lambda}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + \frac{8}{3} - \aleph(z) \right]. \quad (\text{B.8})$$

For some frequently occurring combinations of special functions we applied the following abbreviations

$$\zeta'(x, y) \equiv \partial_x \zeta(x, y), \quad (\text{B.9})$$

$$\aleph(n, z) \equiv \zeta'(-n, z) + (-1)^{n+1} \zeta'(-n, z^*), \quad (\text{B.10})$$

$$\aleph(z) \equiv \Psi(z) + \Psi(z^*), \quad (\text{B.11})$$

where  $n$  is assumed to be an integer and  $z$  a general complex number, here  $z = 1/2 - i\hat{\mu}$ . Here  $\zeta$  denotes the Riemann zeta function,  $\Psi$  is the digamma function,

$$\Psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)}. \quad (\text{B.12})$$

Below we enlist the values of the function  $\aleph$  as required for our calculation. Though the following list are given at small values of  $\mu/T$ , in the actual plot we calculate  $\aleph$  for any value of  $\mu$  using *Mathematica*.

$$\aleph(z) = -2\gamma_E - 4\ln 2 + 14\zeta(3)\hat{\mu}^2 - 62\zeta(5)\hat{\mu}^4 + 127\zeta(7)\hat{\mu}^6 + \mathcal{O}(\hat{\mu}^8), \quad (\text{B.13})$$

$$\begin{aligned} \aleph(1, z) = & -\frac{1}{12} \left( \ln 2 - \frac{\zeta'(-1)}{\zeta(-1)} \right) - (1 - 2\ln 2 - \gamma_E) \hat{\mu}^2 - \frac{7}{6} \zeta(3) \hat{\mu}^4 \\ & + \frac{31}{15} \zeta(5) \hat{\mu}^6 + \mathcal{O}(\hat{\mu}^8) \end{aligned} \quad (\text{B.14})$$

## B.2 HTL one-loop sum-integrals

We also need some more difficult one-loop sum-integrals that involve the angular average defined earlier in Eq 6.9. We list them below.

1.

$$\oint_{\{P\}} \frac{1}{P^4} \mathcal{T}_P = \frac{d-4}{d-5} \oint_{\{P\}} \frac{1}{P^4}. \quad (\text{B.15})$$

2.

$$\begin{aligned} \oint_{\{P\}} \frac{1}{p^2 P^2} \mathcal{T}_P &= \left\langle \frac{1 - c^{4-d}}{1 - c^2} \right\rangle_c \oint_{\{P\}} \frac{1}{p^2 P^2}, \\ &= - \left\langle \frac{1 - c^{4-d}}{1 - c^2} \right\rangle_c \frac{2}{d-2} \oint_{\{P\}} \frac{1}{P^4}, \\ &= - \frac{2A_1}{d-2} \oint_{\{P\}} \frac{1}{P^4}, \end{aligned} \quad (\text{B.16})$$

with

$$A_1 = \left\langle \frac{1 - c^{4-d}}{1 - c^2} \right\rangle_c. \quad (\text{B.17})$$

3.

$$\begin{aligned} \oint_{\{P\}} \frac{1}{p^2 P^2} \mathcal{T}_P^2 &= \oint_{\{P\}} \left\langle \frac{1}{p^2 P^2} \left( \frac{p_0^2}{p_0^2 - p^2 c_1^2} \right) \left( \frac{p_0^2}{p_0^2 - p^2 c_2^2} \right) \right\rangle_{c_1, c_2}, \\ &= \oint_{\{P\}} \left\langle \frac{p_0^4}{p^2 P^2} \left[ \frac{1}{p_0^2 - p^2 c_1^2} - \frac{1}{p_0^2 - p^2 c_2^2} \right] \frac{1}{p^2 (c_1^2 - c_2^2)} \right\rangle_{c_1, c_2} \\ &= \oint_{\{P\}} \left\langle \frac{p_0^4}{p^4 (c_1^2 - c_2^2)} \left[ \left( \frac{1}{P^2} - \frac{1}{p_0^2 - p^2 c_1^2} \right) \frac{1}{p^2 (1 - c_1^2)} - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2} \\ &= \oint_{\{P\}} \left\langle \frac{p_0^4}{p^6 (c_1^2 - c_2^2)} \left[ \left( \frac{1 - c_1^{3+2\epsilon}}{P^2} \right) \frac{1}{1 - c_1^2} - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2} \end{aligned}$$



$$\begin{aligned}
&= \sum_{\{P\}} \frac{p_0^4}{p^6 P^2} \left\langle \frac{1 - c_1^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left( \frac{2}{p^4} + \frac{P^2}{p^6} + \frac{1}{p^2 P^2} \right) \left\langle \frac{1 - c_1^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \frac{1}{p^2 P^2} \left\langle \frac{1 - c_1^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= -\frac{2A_2}{d-2} \sum_{\{P\}} \frac{1}{P^4}, \tag{B.18}
\end{aligned}$$

with

$$A_2 = \left\langle \frac{1 - c_1^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2}. \tag{B.19}$$

4.

$$\begin{aligned}
\sum_{\{P\}} \frac{1}{p_0^2 P^2} \mathcal{T}_P^2 &= \sum_{\{P\}} \left\langle \frac{1}{p_0^2 P^2} \left( \frac{p_0^2}{p_0^2 - p^2 c_1^2} \right) \left( \frac{p_0^2}{p_0^2 - p^2 c_2^2} \right) \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left\langle \frac{p_0^2}{P^2} \left[ \frac{1}{p_0^2 - p^2 c_1^2} - \frac{1}{p_0^2 - p^2 c_2^2} \right] \frac{1}{p^2 (c_1^2 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left\langle \frac{p_0^2}{p^2 (c_1^2 - c_2^2)} \left[ \left( \frac{1}{P^2} - \frac{1}{p_0^2 - p^2 c_1^2} \right) \frac{1}{p^2 (1 - c_1^2)} - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left\langle \frac{p_0^2}{p^4 (c_1^2 - c_2^2)} \left[ \left( \frac{1 - c_1^{1+2\epsilon}}{P^2} \right) \frac{1}{1 - c_1^2} - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \frac{p_0^2}{p^4 P^2} \left\langle \frac{1 - c_1^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left( \frac{1}{p^4} + \frac{1}{p^2 P^2} \right) \left\langle \frac{1 - c_1^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \frac{1}{p^2 P^2} \left\langle \frac{1 - c_1^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= -\frac{2A_3}{d-2} \sum_{\{P\}} \frac{1}{P^4}, \tag{B.20}
\end{aligned}$$

with

$$A_3 = \left\langle \frac{1 - c_1^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2}. \tag{B.21}$$

5.

$$\begin{aligned}
\sum_{\{P\}} \frac{1}{P^4} \mathcal{T}_P^2 &= \sum_{\{P\}} \left\langle \frac{1}{P^4} \left( \frac{p_0^2}{p_0^2 - p^2 c_1^2} \right) \left( \frac{p_0^2}{p_0^2 - p^2 c_2^2} \right) \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left\langle \frac{p_0^4}{p^2 P^4 (c_1^2 - c_2^2)} \left[ \frac{1}{p_0^2 - p^2 c_1^2} - \frac{1}{p_0^2 - p^2 c_2^2} \right] \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left\langle \frac{p_0^4}{p^2 P^2 (c_1^2 - c_2^2)} \left[ \left( \frac{1}{P^2} - \frac{1}{p_0^2 - p^2 c_1^2} \right) \frac{1}{p^2 (1 - c_1^2)} - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left\langle \frac{p_0^4}{p^4 P^4 (c_1^2 - c_2^2)} \left( \frac{1}{(1 - c_1^2)} - \frac{1}{(1 - c_2^2)} \right) \right\rangle_{c_1, c_2} \\
&\quad - \sum_{\{P\}} \left\langle \frac{p_0^4}{p^2 (c_1^2 - c_2^2)} \left[ \left( \frac{1}{P^2} - \frac{1}{p_0^2 - p^2 c_1^2} \right) \frac{1}{p^4 (1 - c_1^2)^2} - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \frac{p_0^4}{p^4 P^4} \left\langle \frac{1}{(1 - c_1^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} - \sum_{\{P\}} \left\langle \frac{p_0^4}{p^6 P^2 (c_1^2 - c_2^2)} \left[ \frac{1 - c_1^{3+2\epsilon}}{(1 - c_1^2)^2} - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left( \frac{1}{p^4} + \frac{1}{P^4} + \frac{2}{p^2 P^2} \right) \left\langle \frac{1}{(1 - c_1^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&\quad - \sum_{\{P\}} \left( \frac{1}{p^2 P^2} + \frac{P^2}{p^6} + \frac{2}{p^4} \right) \left\langle \left[ \frac{1 - c_1^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{1 - c_2^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right] \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left( \frac{1}{P^4} + \frac{1}{p^2 P^2} \right) \left\langle \frac{1}{(1 - c_1^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&\quad + \sum_{\{P\}} \frac{1}{p^2 P^2} \left\langle \left[ \frac{c_1^{3+2\epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{3+2\epsilon} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right] \right\rangle_{c_1, c_2} \\
&= \left( \frac{d-4}{d-2} A_4 - \frac{2}{d-2} A_5 \right) \sum_{\{P\}} \frac{1}{P^4}, \tag{B.22}
\end{aligned}$$

with

$$A_4 = \left\langle \frac{1}{(1 - c_1^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \tag{B.23}$$

$$A_5 = \left\langle \left[ \frac{c_1^{3+2\epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{3+2\epsilon} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right] \right\rangle_{c_1, c_2}. \tag{B.24}$$

6.

$$\sum_{\{P\}} \frac{p_3^2}{p_0^2 P^4} \mathcal{T}_P^2 = \sum_{\{P\}} \frac{p_3^2 p_0^2}{p^4 P^2} \left\langle \frac{1}{c_1^2 - c_2^2} \left[ \frac{1}{1 - c_1^2} \left( \frac{1}{P^2} - \frac{1}{p_0^2 - p^2 c_1^2} \right) - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2}$$

$$\begin{aligned}
&= \sum_{\{P\}} \left( \frac{p_3^2}{p^2 P^4} + \frac{p_3^2}{p^4 P^2} \right) \left\langle \frac{1}{(1-c_1^2)(1-c_2^2)} \right\rangle_{c_1, c_2} \\
&\quad - \sum_{\{P\}} \frac{p_3^2 p_0^2}{p^6} \left\langle \frac{1}{(c_1^2 - c_2^2)} \left[ \frac{1}{(1-c_1^2)^2} \left( \frac{1}{P^2} - \frac{1}{p_0^2 - p^2 c_1^2} \right) - c_1 \leftrightarrow c_2 \right] \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \left( \frac{p_3^2}{p^2 P^4} + \frac{p_3^2}{p^4 P^2} \right) \left\langle \frac{1}{(1-c_1^2)(1-c_2^2)} \right\rangle_{c_1, c_2} \\
&\quad - \sum_{\{P\}} \frac{p_3^2}{p^4 P^2} \left\langle \frac{1 - c_1^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{1 - c_2^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right\rangle_{c_1, c_2} \\
&= \sum_{\{P\}} \frac{p_3^2}{p^2 P^4} \left\langle \frac{1}{(1-c_1^2)(1-c_2^2)} \right\rangle_{c_1, c_2} \\
&\quad + \sum_{\{P\}} \frac{p_3^2}{p^4 P^2} \left\langle \frac{c_1^{1+2\epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{1+2\epsilon} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right\rangle_{c_1, c_2} \\
&= \left( \frac{A_4}{d} - \frac{2}{d(d-2)} A_6 \right) \sum_{\{P\}} \frac{1}{P^4}, \tag{B.25}
\end{aligned}$$

with

$$A_6 = \left\langle \frac{c_1^{1+2\epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{1+2\epsilon} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right\rangle_{c_1, c_2} \tag{B.26}$$

In the following section, we calculate the angular average of various  $A_i$ 's that arose in the above equations.

### B.3 c-integrations

1. From Eq. (B.17), we can write

$$A_1 = \left\langle \frac{1 - c^{4-d}}{1 - c^2} \right\rangle_c = 1 + \frac{1}{d-3} - \frac{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right) \sec\left(\frac{d\pi}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}. \tag{B.27}$$

2. From Eq. (B.19), we can write

$$\begin{aligned}
A_2 &= \left\langle \frac{1 - c_1^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{3+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= -\frac{\pi^2}{12} + 2 \log 2 + \left( \frac{\pi^2}{3} - 2 \log 2 (2 - \log 2) - \frac{\zeta(3)}{2} \right) \epsilon + \mathcal{O}(\epsilon^2). \tag{B.28}
\end{aligned}$$

3. From Eq. (B.21), we can write

$$\begin{aligned}
A_3 &= \left\langle \frac{1 - c_1^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_1^2)} - \frac{1 - c_2^{1+2\epsilon}}{(c_1^2 - c_2^2)(1 - c_2^2)} \right\rangle_{c_1, c_2} \\
&= -\frac{\pi^2}{12} + \left( \frac{\pi^2}{3} - \frac{\zeta(3)}{2} \right) \epsilon + \mathcal{O}(\epsilon^2). \tag{B.29}
\end{aligned}$$

4. From Eq. (B.23), we can write

$$A_4 = \left\langle \frac{1}{(1-c_1^2)(1-c_2^2)} \right\rangle_{c_1, c_2} = \frac{(d-2)^2}{(d-3)^2} = \frac{1}{4\epsilon^2} - \frac{1}{\epsilon} + 1. \quad (\text{B.30})$$

5. From Eq. (B.24), we can write

$$A_5 = \left\langle \frac{c_1^{3+2\epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{3+2\epsilon} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right\rangle_{c_1, c_2}. \quad (\text{B.31})$$

Unlike other  $A_i$  functions, computation of  $A_5$  is not straightforward and can not be done directly analytically in *Mathematica* in  $\epsilon \rightarrow 0$ . After calculating the angular average in Eq. (B.31), we end up with the following equation.

$$\begin{aligned} A_5 = & -\frac{1}{\pi\Gamma(1-\epsilon)^2}\Gamma(-1-\epsilon)^2\Gamma(3/2-\epsilon)^2 \\ & \times \left[ \frac{1}{3}4^{-1-\epsilon}e^{-i\pi\epsilon}(1+3\epsilon+2\epsilon^2)\Gamma(1+2\epsilon)\left\{ 3e^{2i\pi\epsilon}F\left(\epsilon, \frac{3}{2}+\epsilon \middle| \frac{1}{2}\right) \right. \right. \\ & + (2+e^{2i\pi\epsilon})\left( 3F\left(\frac{3}{2}+\epsilon, 2+\epsilon \middle| \frac{1}{2}\right) - 6(3+2\epsilon)F\left(2+\epsilon, \frac{5}{2}+\epsilon \middle| \frac{3}{2}\right) \right. \\ & + (5+2\epsilon)(3+2\epsilon)F\left(2+\epsilon, \frac{7}{2}+\epsilon \middle| \frac{5}{2}\right) \left. \left. \right\} - \frac{\pi}{\Gamma[-1/2-\epsilon]^2}\left\{ -F\left(1, \frac{3}{2}+\epsilon \middle| -\frac{1}{2}-\epsilon\right) \right. \right. \\ & \left. \left. - \frac{3+2\epsilon}{1+2\epsilon}F\left(1, \frac{5}{2}+\epsilon \middle| \frac{1}{2}-\epsilon\right) + \frac{1}{(1+2\epsilon)^2}F\left(1, \frac{3}{2}, \frac{1}{2}+\epsilon \middle| -\frac{1}{2}, \frac{1}{2}-\epsilon\right) + \frac{3}{1-4\epsilon^2}F\left(1, \frac{5}{2}, \frac{3}{2}+\epsilon \middle| \frac{1}{2}, \frac{3}{2}-\epsilon\right) \right\} \right]. \quad (\text{B.32}) \end{aligned}$$

Eq. (B.32) can not be expanded directly at small  $\epsilon$  in *Mathematica*. So, we use the following technique to expand Eq. (B.32) at small  $\epsilon$ . In Eq. (B.32),  $F$  represents the generalized hypergeometric function. The generalized hypergeometric function of type  ${}_pF_q$  is an analytic function of one variable with  $p+q$  parameters. Here, the parameters are functions of  $\epsilon$ , so the list of parameters sometimes gets so lengthy and the standard notation for these functions becomes cumbersome. We therefore introduce a more compact notation as

$$F\left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_n \\ \beta_1, \dots, \beta_{n-1} \end{matrix} \middle| 1\right) = {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \quad (\text{B.33})$$

It is not possible to directly expand  ${}_pF_q$  at small  $\epsilon$ . So, we will use the following procedure to expand  ${}_pF_q$  in the series of  $\epsilon$ .

In Eq. (B.32), there are two types of hypergeometric function *viz.*  ${}_2F_1$  and  ${}_3F_2$ .  ${}_2F_1$  can be expanded in small  $\epsilon$  if one uses the following relation.

$$F(\alpha_1, \alpha_2; \beta_1; 1) = \frac{\Gamma(\beta_1)\Gamma(\beta_1 - \alpha_1 - \alpha_2)}{\Gamma(\beta_1 - \alpha_1)\Gamma(\beta_1 - \alpha_2)} \quad (\text{B.34})$$

To expand  ${}_3F_2$  type of hypergeometric function, we can try the following power series representation for the generalized hypergeometric function as

$$F \left( \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n n!} z^n, \quad (\text{B.35})$$

where  $(a)_b$  is Pochhammer's symbol:

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}. \quad (\text{B.36})$$

The power series converges for  $|z| < 1$ . For  $z = 1$ , it converges if  $\text{Re } s > 0$ , where

$$s = \sum_{i=1}^{p-1} \beta_i - \sum_{i=1}^p \alpha_i. \quad (\text{B.37})$$

In Eq. (B.32), both  ${}_3F_2$  has negative  $s$  value for  $\epsilon \rightarrow 0$ . So, we will use the following relation to change the parameters to make  $s$  value positive.

$$F \left( \begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \middle| 1 \right) = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(s)}{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\alpha_3)} F \left( \begin{matrix} \beta_1 - \alpha_3, \beta_2 - \alpha_3, s \\ \alpha_1 + s, \alpha_2 + s \end{matrix} \middle| 1 \right) \quad (\text{B.38})$$

• **Expansion of  ${}_3F_2(1, \frac{3}{2}, \frac{1}{2} + \epsilon; -\frac{1}{2}, \frac{1}{2} - \epsilon; 1)$**

Here,  $s = \beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3 = -3 - 2\epsilon < 0$  at  $\epsilon \rightarrow 0$ . So, we use Eq. (B.38) to change the parameter.

$$F \left( \begin{matrix} 1, \frac{3}{2}, \frac{1}{2} + \epsilon \\ -\frac{1}{2}, \frac{1}{2} - \epsilon \end{matrix} \middle| 1 \right) = \frac{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2} - \epsilon)\Gamma(3 - 2\epsilon)}{\Gamma(-2 - 2\epsilon)\Gamma(-\frac{3}{2} - 2\epsilon)\Gamma(\frac{1}{2} + \epsilon)} F \left( \begin{matrix} -1 - \epsilon, -2\epsilon, -3 - 2\epsilon \\ -2 - 2\epsilon, -\frac{3}{2} - 2\epsilon \end{matrix} \middle| 1 \right) \quad (\text{B.39})$$

For the hypergeometric function that appears in RHS of the above eqn,  $s = 1/2 + 9\epsilon > 0$  at  $\epsilon \rightarrow 0$ , so the power series expansion will converge.

So,

$$\begin{aligned} & F \left( \begin{matrix} -1 - \epsilon, -2\epsilon, -3 - 2\epsilon \\ -2 + 2\epsilon, -\frac{3}{2} + 2\epsilon \end{matrix} \middle| 1 \right) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(-1 - \epsilon + n)\Gamma(-2\epsilon + n)\Gamma(-3 - 2\epsilon + n)\Gamma(-2 - 2\epsilon)\Gamma(-\frac{3}{2} - 2\epsilon)}{\Gamma(-2 - 2\epsilon + n)\Gamma(-\frac{3}{2} - 2\epsilon + n)\Gamma(-1 - \epsilon)\Gamma(-2\epsilon)\Gamma(-3 - 2\epsilon)} \frac{1}{n!}. \end{aligned} \quad (\text{B.40})$$

In the above sum, if one expand the term within summation at small  $\epsilon$ , one ends up with an expression that diverges for  $n = 0$  to  $n = 3$ . To avoid that we will do the summation before the expansion from  $n = 0$  to  $n = 3$ . From  $n = 4$ , we will perform the summation after expansion in small  $\epsilon$ .

So,

$${}_3F_2 \left( -1 - \epsilon, -2\epsilon, -3 - 2\epsilon; -2 + 2\epsilon, -\frac{3}{2} + 2\epsilon; 1 \right)$$

$$\begin{aligned}
&= \left( \sum_{n=0}^3 + \sum_{n=4}^{\infty} \right) \frac{\Gamma(-1-\epsilon+n) \Gamma(-2\epsilon+n) \Gamma(-3-2\epsilon+n) \Gamma(-2-2\epsilon) \Gamma(-\frac{3}{2}-2\epsilon)}{\Gamma(-2-2\epsilon+n) \Gamma(-\frac{3}{2}-2\epsilon+n) \Gamma(-1-\epsilon) \Gamma(-2\epsilon) \Gamma(-3-2\epsilon)} \frac{1}{n!} \\
&= \left[ 1 - \frac{14}{3}\epsilon + \frac{64}{9}\epsilon^2 - \frac{652}{27}\epsilon^3 + \mathcal{O}(\epsilon^4) \right] + \sum_{n=4}^{\infty} \left[ \frac{8\sqrt{\pi} \Gamma(n-1)}{n(n-3)\Gamma(n-\frac{3}{2})} \epsilon^2 \right. \\
&\quad \left. - \frac{8\sqrt{\pi} \Gamma(n) \left( 3\gamma_E + 11 + \frac{9}{n-3} + \frac{9}{n-2} + \frac{6}{n-1} + 15\psi^{(0)}(n-3) - 12\psi^{(0)}(2n-4) \right) \epsilon^3}{3n(n-3)(n-1)\Gamma(n-\frac{3}{2})} + \mathcal{O}(\epsilon^4) \right] \\
&= 1 - \frac{14}{3}\epsilon + \frac{64}{9}\epsilon^2 - \frac{652}{27}\epsilon^3 + \frac{16}{3} {}_4F_3 \left( 1, 1, 3, 4; 2, 5, \frac{5}{2}; 1 \right) \epsilon^2 \\
&\quad + \frac{1}{18} \left[ 32(3\gamma_E - 11 + 6\log 2) {}_3F_2 \left( 1, 1, 4; \frac{5}{2}, 5; 1 \right) + 32(6\gamma_E - 11 + 12\log 2) {}_3F_2 \left( 1, 1, 3; \frac{3}{2}, 4; 1 \right) \right. \\
&\quad \left. + 252\zeta(3) - 36\gamma_E\pi^2 - 51\pi^2 - 640\gamma_E + 824 - 72\pi^2 \log 2 - 1280 \log 2 \right] \epsilon^3 + \mathcal{O}(\epsilon^4) \quad (\text{B.41})
\end{aligned}$$

The hypergeometric function of type  ${}_{p+1}F_{q+1}$  has an integral representation in terms of the hypergeometric function of type  ${}_pF_q$ :

$$\int_0^1 dt t^{\nu-1} (1-t)^{\mu-1} F \left( \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \middle| tz \right) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} F \left( \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p, \nu \\ \beta_1, \dots, \beta_q, \mu+\nu \end{matrix} \middle| z \right). \quad (\text{B.42})$$

Using Eq. (B.42), Eq. (B.41) can be simplified as

$$\begin{aligned}
&{}_3F_2 \left( -1-\epsilon, -2\epsilon, -3-2\epsilon; -2+2\epsilon, -\frac{3}{2}+2\epsilon; 1 \right) \\
&= 1 - \frac{14}{3}\epsilon + \frac{64}{9}\epsilon^2 - \frac{652}{27}\epsilon^3 + \frac{16}{3} \int_0^1 dt F \left( \begin{matrix} 1, 3, 4 \\ 5, \frac{5}{2} \end{matrix} \middle| t \right) \epsilon^2 \\
&\quad + \frac{1}{18} \left[ 32(3\gamma_E - 11 + 6\log 2) \times 3 \int_0^1 dt t^2 {}_2F_1 \left( 1, 1; \frac{3}{2}; t \right) \right. \\
&\quad \left. + 32(6\gamma_E - 11 + 12\log 2) \times 4 \int_0^1 dt t^3 {}_2F_1 \left( 1, 1; \frac{5}{2}; t \right) \right. \\
&\quad \left. + 252\zeta(3) + 824 - 51\pi^2 - 4(160 + 9\pi^2)(\gamma_E + 2\log 2) \right] \epsilon^3 + \mathcal{O}(\epsilon^4) \\
&= 1 - \frac{14}{3}\epsilon + \frac{64}{9}\epsilon^2 - \frac{652}{27}\epsilon^3 + \left( \pi^2 + \frac{112}{9} \right) \epsilon^2 + \left( 14\zeta(3) - \frac{2}{27} (746 + 63\pi^2) \right) \epsilon^3 + \mathcal{O}(\epsilon^4) \\
&= 1 - \frac{14}{3}\epsilon + \left( \pi^2 + \frac{176}{9} \right) \epsilon^2 + \left( -\frac{2144}{27} - \frac{14\pi^2}{3} + 14\zeta(3) \right) \epsilon^3 + \mathcal{O}(\epsilon^4) \quad (\text{B.43})
\end{aligned}$$

So, Eq. (B.39) can be rewritten as

$$F \left( 1, \frac{3}{2}, \frac{1}{2} + \epsilon; -\frac{1}{2}, \frac{1}{2} - \epsilon; 1 \right) = \frac{\Gamma(-\frac{1}{2}) \Gamma(\frac{1}{2} - \epsilon) \Gamma(3 - 2\epsilon)}{\Gamma(-2 - 2\epsilon) \Gamma(-\frac{3}{2} - 2\epsilon) \Gamma(\frac{1}{2} + \epsilon)}$$

$$\times \left[ 1 - \frac{14}{3}\epsilon + \left( \pi^2 + \frac{176}{9} \right) \epsilon^2 + \left( -\frac{2144}{27} - \frac{14\pi^2}{3} + 14\zeta(3) \right) \epsilon^3 + \mathcal{O}(\epsilon^4) \right] \quad (\text{B.44})$$

• **Expansion of  ${}_3F_2 \left( 1, \frac{5}{2}, \frac{3}{2} + \epsilon; \frac{1}{2}, \frac{3}{2} - \epsilon; 1 \right)$**

Following the similar procedure, we can write

$${}_3F_2 \left( 1, \frac{5}{2}, \frac{3}{2} + \epsilon; \frac{1}{2}, \frac{3}{2} - \epsilon; 1 \right) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2} - \epsilon\right) \Gamma(3 - 2\epsilon)}{\Gamma(-2 - 2\epsilon) \Gamma\left(-\frac{1}{2} - 2\epsilon\right) \Gamma\left(\frac{3}{2} + \epsilon\right)} \\ \times \left[ 1 - \frac{10}{3}\epsilon + \left( \pi^2 + \frac{40}{3} \right) \epsilon^2 + \left( -\frac{160}{3} - \frac{10\pi^2}{3} + 14\zeta(3) \right) \epsilon^3 + \mathcal{O}(\epsilon^4) \right] \quad (\text{B.45})$$

Adding all the contribution, we can write Eq. (B.32) as

$$A_5 = \left\langle \frac{c_1^{3+2\epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{3+2\epsilon} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right\rangle_{c_1, c_2} \\ = -\frac{1}{8\epsilon^2} + \frac{1}{4\epsilon} + \frac{1}{24} (18 - \pi^2) + \frac{1}{12} (2\pi^2 - 3(9 + \zeta(3)))\epsilon + \mathcal{O}(\epsilon^2) \quad (\text{B.46})$$

6. The remaining  $A$ - function  $A_6$  can be written from Eq. (B.24) as

$$A_6 = \left\langle \frac{c_1^{1+2\epsilon} - c_1^2}{(c_1^2 - c_2^2)(1 - c_1^2)^2} - \frac{c_2^{1+2\epsilon} - c_2^2}{(c_1^2 - c_2^2)(1 - c_2^2)^2} \right\rangle_{c_1, c_2} \\ = A_4 + A_5 - A_3 \\ = \frac{1}{8\epsilon^2} - \frac{3}{4\epsilon} + \frac{1}{24} (\pi^2 + 42) + \frac{1}{12} (3(\zeta(3) - 9) - 2\pi^2)\epsilon + \mathcal{O}(\epsilon^2) \quad (\text{B.47})$$

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